MATH4250 Game Theory
Exercise 5

Assignment 5: 1, 2, 3, 4, 5 (Due: 18 April 2019 (Wednesday))

1. Let $A = \{A_1, A_2, A_3\}$ be the player set and $X_i = \{0, 1\}$, for $i = 1, 2, 3$, be the strategy set for A_i. Suppose the payoffs to the players are given by the following table.

Strategy	Payoff vector
$(0, 0, 0)$	$(-2, 3, 5)$
$(0, 0, 1)$	$(1, -2, 7)$
$(0, 1, 0)$	$(1, 5, 0)$
$(0, 1, 1)$	$(10, -3, -1)$
$(1, 0, 0)$	$(-1, 0, 7)$
$(1, 0, 1)$	$(-4, 4, 6)$
$(1, 1, 0)$	$(12, -4, -2)$
$(1, 1, 1)$	$(-1, 5, 2)$

   (a) Find the characteristic function of the game.

   (b) Show that the core of the game is empty.

2. Consider a three-person game with characteristic function

$$\begin{aligned} \nu(\{1\}) &= 27 \\ \nu(\{2\}) &= 8 \\ \nu(\{3\}) &= 18 \\ \nu(\{1,2\}) &= 36 \\ \nu(\{1,3\}) &= 50 \\ \nu(\{2,3\}) &= 27 \\ \nu(\{1,2,3\}) &= 60 \end{aligned}$$

   Find the core of the game and draw the region representing the core on the $x_1 - x_2$ plane.

3. Let ν be the characteristic function defined by $\nu(\{1\}) = 3, \nu(\{2\}) = 4, \nu(\{3\}) = 6, \nu(\{1,2\}) = 9, \nu(\{1,3\}) = 12, \nu(\{2,3\}) = 15, \nu(\{1,2,3\}) = 20$.

   (a) Let μ be the $(0,1)$ reduced form of ν. Find $\mu(\{1,2\}), \mu(\{1,3\}), \mu(\{2,3\})$.

   (b) Find the core of ν and draw the region representing the core on the $x_1 - x_2$ plane.

   (c) Find the Shapley values of the players.

4. Three towns A, B, C are considering whether to built a joint water distribution system. The costs of the construction works are listed in the following table

Coalition	Cost(in million dollars)
$\{A\}$	11
$\{B\}$	7
$\{C\}$	8
$\{A,B\}$	15
$\{A,C\}$	14
$\{B,C\}$	13
$\{A,B,C\}$	20

For any coalition $S \subset \{A,B,C\}$, define $\nu(S)$ to be the amount saved if they build the system together. Find the Shapley values of A,B,C and the amount that each of them should pay if they cooperate.

5. Players 1, 2, 3 and 4 have 45, 25, 15, and 15 votes respectively. In order to pass a certain resolution, 51 votes are required. For any coalition S, define $\nu(S) = 1$ if S can pass a certain resolution. Otherwise $\nu(S) = 0$. Find the Shapley values of the players.

6. Players 1, 2, 3 and 4 have 40, 30, 20, and 10 shares if stocks respectively. In order to pass a certain decision, 50 shares are required. For any coalition S, define $\nu(S) = 1$ if S can pass a certain decision. Otherwise $\nu(S) = 0$. Find the Shapley values of the players.

7. Consider the following market game. Each of the 5 players starts with one glove. Two of them have a right-handed glove and three of them have a left-handed glove. At the end of the game, an assembled pair is worth \$1 to whoever holds it. Find the Shapley value of the players.

8. Let $\mathcal{A} = \{1,2,3\}$ be the set of players and ν be a game in characteristic form with

$$\begin{aligned}
\nu(\{1\}) &= -a \\
\nu(\{2\}) &= -b \\
\nu(\{3\}) &= -c \\
\nu(\{2,3\}) &= a \\
\nu(\{1,3\}) &= b \\
\nu(\{1,2\}) &= c \\
\nu(\{1,2,3\}) &= 1
\end{aligned}$$

where $0 \le a,b,c \le 1$.

(a) Let μ be the $(0,1)$ reduced form of ν. Find $\mu(\{1,2\})$, $\mu(\{1,3\})$, $\mu(\{2,3\})$ in terms of a,b,c.

(b) Suppose $a+b+c=2$. Find an imputation $\mathbf{x}$ of ν which lies in the core $C(\nu)$ in terms of a,b,c and prove that $C(\nu) = \{\mathbf{x}\}$.

9. Consider an airport game which is a cost allocation problem. Let $N = \{1,2,\cdots,n\}$ be the set of players. For each $i = 1,2,\cdots,n$, player i requires an airfield that costs c_i to build. To accommodate all the players, the field will be built at a cost

of $\max_{1 \le i \le n} c_i$. Suppose all the costs are distinct and $c_1 < c_2 < \cdots < c_n$. Take the characteristic function of the game to be

$$\nu(S) = -\max_{i \in S} c_i$$

For each $k = 1, 2, \cdots, n$, let $R_k = \{k, k+1, \cdots, n\}$ and define

$$\nu_k(S) = \begin{cases} -(c_k - c_{k-1}) & \text{if } S \cap R_k \neq \emptyset \\ 0 & \text{if } S \cap R_k = \emptyset \end{cases}$$

(a) Show that $\nu = \sum_{k=1}^{n} \nu_k$

(b) Show that for each $k = 1, 2, \cdots, n$, if $i \notin R_k$, then player i is a null player of ν_k.

(c) Show that for each $k = 1, 2, \cdots, n$, if $i, j \in R_k$, then player i and player j are symmetric players of ν_k.

(d) Find the Shapley value $\phi_k(\nu)$ of player k, $k = 1, 2, \cdots, n$, of the airport game ν.

10. Let $\mathcal{A} = \{1, 2, \cdots, N\}$. Prove that for any $i \in \mathcal{A}$

$$\sum_{\{i\} \subset S \subset \mathcal{A}} (N - |S|)!(|S| - 1)! = N!$$